Modulational instability of spatially broadband nonlinear optical pulses in four-state atomic systems

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The modulational instability of broadband optical pulses in a four-state atomic system is investigated. In particular, starting from a recently derived generalized nonlinear Schrödinger equation, a wave-kinetic equation is derived. A comparison between coherent and random-phase wave states is made. It is found that the spatial spectral broadening can contribute to the nonlinear stability of ultrashort optical pulses. In practical terms, this could be achieved by using random-phase plate techniques.

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The propagation of ultrashort optical pulses in nonlinear media is of great importance in a wide variety of applications, e.g., fiber optical systems [1,2], atmospheric remote sensing using femtosecond laser pulses in air [3–5], and inertial confinement fusion [6]. The nonlinear propagation of ultrashort optical pulses can be modeled by using the nonlinear Schrödinger equation with short-pulse nonlinear derivative corrections, which can give rise to the filamentation of light pulses, and subsequent formation of light pipes. Modulational and filamentational instabilities, therefore, sometimes pose problems concerning pulse propagation in nonlinear media. Thus, for propagation times much longer than the typical pulse length, it is of importance to find means for the nonlinear stabilization of such optical pulses.

In this Brief Report, we investigate the statistical properties of a generalized nonlinear Schrödinger equation, taking into account the Kerr nonlinearity, linear absorption, nonlinear dispersion, delay in nonlinear refractive index, third-order dispersion, differential absorption, and diffraction. The equation is of relevance for Raman excited four-state atomic systems. We derive an equivalent wave-kinetic equation that governs the propagation of optical quasi-particles, enabling us to study the effect of partial pulse coherence. The newly derived equation is analyzed for the case of spectral pulse broadening, and it is found that the relevant growth rate can

be significantly reduced if appropriate random-phase techniques are used.

Recently, Hang *et al.* [7] derived a generalized nonlinear Schrödinger equation (NLSE) that describes the nonlinear propagation of optical pulses in Raman excited four-state atomic systems (see Fig. 1). In a dimensionless form, the NLSE can be written as

$$i\partial_z u + \partial_t^2 u + 2|u|^2 u = -i[d_0 u - d_1 \partial_t(|u|^2 u) - d_2 u \partial_t |u|^2 - d_3 \partial_t^3 u]$$
$$+ d_4 \partial_t u - d_5 \nabla_\perp^2 u, \tag{1}$$

where u is the normalized pulse amplitude and d_j for $j = 0$ to 5 is the dimensionless coefficients for linear absorption, nonlinear dispersion, delay in nonlinear refractive index, third-order dispersion, differential absorption, and diffraction, respectively. Equation (1) thus describes the nonlinear propagation of ultrashort optical pulses when the terms on the right-hand side of Eq. (1) are significant (i.e., $d_j \sim 1$) [16]. In what follows, we will look at the one-dimensional problem, and thus set $d_5 = 0$.

In order to analyze the spectral evolution of partially coherent optical pulses, we next introduce the Fourier trans-
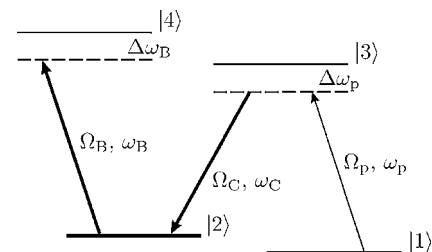
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FIG. 1. Energy-level diagram showing the excitation of the relevant four-state atomic system. Here ω_p is the center frequency of the weak probe field, while ω_B and ω_C are the frequencies of two strong cw control fields. The one-photon detuning frequencies are defined according to $\Delta\omega_p = \omega_{31} - \omega_p$ and $\Delta\omega_B = \omega_{42} - \omega_B$. Moreover, $2\Omega_p$, $2\Omega_B$, and $2\Omega_C$ are the Rabi frequencies for the indicated transitions, and two-photon resonance is assumed to always be maintained [7,8].

form of the two-point correlation function, i.e., the Wigner function [9–11]

$$\rho(z,t,\omega) = \frac{1}{2\pi} \int d\tau e^{-i\omega\tau} \langle u^*(z,t+\tau/2)u(z,t-\tau/2) \rangle, \quad (2)$$

where the angular bracket denotes the ensemble average [12]. The Wigner function represents a generalized distribution function for optical quasiparticles. The Wigner method [13], as well as the equivalent mutual coherence method [14], have been used to analyze the modulational instability in the "standard" nonlinear Schrödinger picture, in which the right-hand side of Eq. (1) is identically zero. From the definition (2) follows the normalization

$$I(z,t) \equiv \langle |u(z,t)|^2 \rangle = \int d\omega \rho(z,t,\omega), \quad (3)$$

giving the pulse intensity in terms of the Wigner function.

Applying the z derivative to the definition (2), we obtain the wave-kinetic equation for optical quasiparticles,

$$\partial_z \rho - 2\omega \partial_t \rho - 4I \sin\left(\frac{1}{2}\overleftarrow{\partial}_t \overrightarrow{\partial}_\omega\right)\rho$$

$$= -2d_0\rho - d_1\left\{2\omega I \sin\left(\frac{1}{2}\overleftarrow{\partial}_t \overrightarrow{\partial}_\omega\right)\rho - \partial_t\left[I \cos\left(\frac{1}{2}\overleftarrow{\partial}_t \overrightarrow{\partial}_\omega\right)\rho\right]\right\}$$

$$+ 2d_2(\partial_t I)\cos\left(\frac{1}{2}\overleftarrow{\partial}_t \overrightarrow{\partial}_\omega\right)\rho - d_3\left(3\omega^2 - \frac{1}{4}\partial_t^2\right)\partial_t\rho - 2d_4\omega\rho. \quad (4)$$

The wave-kinetic equation (4) determines the phase-space evolution of partially coherent optical wave packets. In the low-frequency limit, we can retain only the first term in the operator expansions to obtain

The case of higher-order dispersive corrections to the nonlinear Schrödinger equation in the Wigner picture has been treated in the wave-number domain in Ref. [15].

In the case of coherent optical pulses, we let $\rho_0(\omega) = I_0\delta(\omega-\omega_0)$. Here the dispersion relation (7) reduces to

$$K = -\left[2\omega_0 + (2d_1+d_2)\bar{I} - d_3(3\omega_0^2 + \Omega^2)\right]\Omega \pm \left[(d_1+d_2)^2\bar{I}^2\Omega^2\right.$$

$$\left. + (1-3d_3\omega_0)^2\Omega^4 - 2(2-d_1\omega_0)(1-3d_3\omega_0)\bar{I}\Omega^2\right]^{1/2}. \quad (8)$$

For simplicity, we focus on the case $\omega_0=0$, at which we have the growth rate $\Gamma=-\text{Im}(K)$,

$$\partial_z \rho - 2\omega \partial_t \rho - 2\partial_t I \partial_\omega \rho = -2d_0\rho - d_1[\omega\partial_t I \partial_\omega \rho - \partial_t(I\rho)]$$

$$+ 2d_2\rho\partial_t I - 3d_3\omega^2\partial_t\rho - 2d_4\omega\rho. \quad (5)$$

Here we see that the terms on the left-hand side of Eq. (5) resemble the terms in a Vlasov equation. Moreover, Eqs. (4) and (5) have a time-independent solution,

$$\bar{\rho}(z,\omega) = \rho_0(\omega)e^{-2(d_0+d_4\omega)z}, \quad (6)$$

exhibiting a spatial diffusive influence of the linear and differential absorption. For a coherent distribution, i.e., $\rho_0(\omega) = I_0\delta(\omega-\omega_0)$ for some frequency ω_0, we obtain the intensity $\bar{I}(z) = I_0\exp[-2(d_0+d_4\omega_0)z]$, while for a Gaussian distribution $\rho_0(\omega) = (I_0/\sqrt{2\pi}\Delta)\exp[-(\omega-\omega_0)^2/2\Delta^2]$ with the spectral width Δ, we obtain the intensity $\bar{I}(z) = I_0\exp[-2(d_0+d_4\omega_0 - d_4\Delta^2 z)z]$. We note that the finite spectral width in the latter case gives rise to *growing* behavior after a certain distance of propagation. In fact, the spatial intensity decay halts for $z_{\text{crit}} = (d_0+d_4\omega_0)/d_4\Delta^2$, indicating the breakdown of the above model, as we do not expect growing modes from loss terms. For a broad spectral distribution, this distance may be short.

Next, we analyze the modulational instability of broadband optical pulses that are governed by Eq. (4). Letting $\rho(z,t,\omega) = \bar{\rho}(z,\omega) + \rho_1(z,\omega)\exp(iKz-i\Omega t)$ in the latter, where $|\rho_1| \ll \bar{\rho}$, we linearize the wave kinetic equation against ρ_1. If the dimensionless perturbation wave number satisfies $K \gg d_0, d_4\omega$, the perturbation wavelength is much smaller than the decay length. Thus, we may treat the background distribution as a constant, and neglect the terms containing d_0 and $d_4\omega$. We then obtain the nonlinear dispersion relation

$$1 = \int d\omega \frac{[2-d_1(\omega+\Omega/2)-d_2\Omega]\bar{\rho}(z,\omega-\Omega/2) - [2-d_1(\omega+\Omega/2)+d_2\Omega]\bar{\rho}(z,\omega+\Omega/2)}{K+2\omega\Omega+d_1\Omega\bar{I}-d_3(3\omega^2+\Omega^2/4)\Omega}. \quad (7)$$

$$\Gamma = [4\bar{I}\Omega^2 - \Omega^4 - (d_1+d_2)^2\bar{I}^2\Omega^2]^{1/2}, \quad (9)$$

where the nonlinear dispersion and delay in the nonlinear refractive index gives a stabilizing effect to the regular Kerr modulational instability (see Fig. 2). We note that the modulational instability follows as a consequence of the nonlinear pulse propagation equation (1) governed by the underlying dyamics of the four-state atomic system [8]. The latter system is in the form of a four-wave mixing set of equations, and it is therefore perhaps not surprising that the modulational instability occurs for the corresponding nonlinear pulse propagation. However, other level configurations may in principle possess similar instability properties, but the issue of the uniqueness of such instabilities to the nonlinear
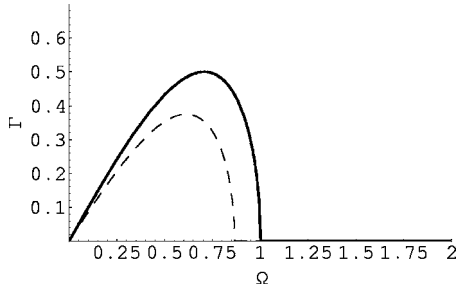
FIG. 2. The coherent modulational instability growth rate Γ plotted as a function of Ω, using Eq. (11) with $\Delta=0$ [see Eq. (9)]. The full thick curve has $d_1=d_2=d_3=d_4=0$, giving the Kerr modulational instability growth rate, while the dashed curve has $d_1=d_2=1$ and $d_3=d_4=0$. We see that the effect of nonlinear dispersion and delay in the nonlinear refractive index is to reduce the modulational instability growth rate.

four-state atomic system is left for future research.

If the background pulse u_0 has a random phase, with a coherence width Δ, this corresponds to a Lorentz distribution

$$\rho_0(\omega) = \frac{I_0}{\pi}\frac{\Delta}{\omega^2 + \Delta^2}.$$
(10)

For the case of $d_4=0$, i.e., vanishing differential absorption, we can integrate the dispersion relation (7) for ρ_0 given by the Lorentz distribution (10) to obtain

$$K = -\left[(2d_1+d_2)\bar{I} - d_3\Omega^2\right]\Omega - 3d_3\Delta^2\Omega + 2i\Delta\Omega$$
$$\pm \left[(d_1+d_2)^2\bar{I}^2\Omega^2 + (1+3id_3\Delta)^2\Omega^4\right.$$
$$\left. - 2(2+id_1\Delta)(1+3id_3\Delta)\bar{I}\Omega^2\right]^{1/2}.$$
(11)

From the dispersion relation (11) we see that the spectral broadening interacts in nontrivial ways with the modifications of the nonlinear Schödinger equation. For a pure Kerr nonlinearity, i.e., $d_j=0$ for all j, the random phase of the background pulse will give rise to a reduction of the modulational instability growth rate [13,14]. Here we find that the random phase will also contribute to the real part of K and the imaginary contribution via interactions between the corrections to the Kerr nonlinearity and spectral broadening. As noted in [15], higher-order dispersive effects may affect the modulational instability growth rate, but on the perturbative level, third-order dispersion only gives a shift in the real wave number. This can be seen by setting $d_1=d_2=0$ and $d_3\Delta, d_3\Omega \ll 1$. Linearizing Eq. (11), we obtain

$$K = d_3\Omega^3 - 3d_3\Delta^2\Omega + 2i\Delta\Omega \pm i(4\bar{I}\Omega^2 - \Omega^4)^{1/2}$$
$$\pm \frac{3d_3\Delta\Omega^2}{2(4\bar{I}\Omega^2 - \Omega^4)^{1/2}}(2\Omega^2 - 4\bar{I}),$$
(12)

and we see that the third-order dispersion only contributes to the real wave-number shift.

We have solved for the modulational instability growth rate numerically and plotted the result for different parameter
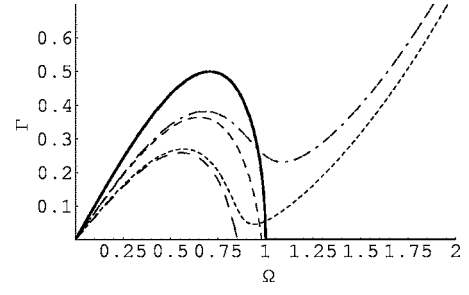

FIG. 3. The incoherent modulational instability growth rate Γ plotted as a function of Ω, using Eq. (11), for different parameter combinations. The full thick curve is the coherent Kerr modulational instability growth rate, i.e., $d_1=d_2=d_3=d_4=\Delta=0$. The remaining curves have $\Delta=0.1$ and $d_4=0$. From the top, the dashed-dotted curve has $d_1=d_2=0$ and $d_3=1$, the dashed curve has $d_1=d_2=d_3=d_4=0$, the dotted curve has $d_1=d_2=d_3=1$, and the curve with long dashes has $d_1=d_2=1$ and $d_3=0$. The strongly growing modes for $\Omega \gtrsim 1$ are, however, not physical, as the underlying assumptions of Eq. (1) are no longer satisfied [see Eq. (12)].

values in Figs. 2 and 3. We note that when $\Delta=0$, we regain the coherent growth rate (9). Moreover, the third-order dispersion only couples to Δ and is thus not present in the coherent case.

From Fig. 3, it is clear that for $d_3=0$, the partial coherence of the background pulse gives rise to a reduced growth rate, and can thus act as a means for stabilizing optical pulses in four-state atomic systems described by Eq. (1). The effect of third-order dispersion is to further reduce the instability growth rate, as expected, and the new branches in the modulational instability growth that occurs for high-frequency perturbations, i.e., $\Omega \gtrsim 1$, are not valid, as the assumptions underlying the derivation of Eq. (1) are no longer satisfied. Thus, the strongly growing modes depicted in Fig. 3 for $\Omega \gtrsim 1$ are not physical.

In conclusion, we have presented an investigation of the modulational instability of broadband optical pulses in Raman excited four-state atomic systems. For this purpose, we have obtained a wave kinetic equation from the modified NLSE, which accounts for numerous nonideal effects embedded on the right-hand side of Eq. (1). Using standard technique, we then derive a nonlinear dispersion relation from the wave kinetic equation. The nonlinear dispersion is analyzed for coherent and broadband spectra of optical pulses. It is found that the growth rate of the modulational instability is reduced in the presence of a Lorentzian optical pulse distribution. A reduced growth rate insures stability of optical pulses over long distances in four-state atomic systems.

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[16] However, we note that the terms proportional to d_j, $j=1,2,3$, on the right-hand side of Eq. (1) are still part of a perturbative expansion, and should be treated as small compared to the terms on the left-hand side. Thus, the time variations in the amplitude u must be weak.